PHASE REVERSAL ACROSS THE SOUND BARRIER

B Shayak

Department of Theoretical and Applied Mechanics,
Sibley School of Mechanical and Aerospace Engineering,
Cornell University,
Ithaca – 14853,
New York, USA

sb2344@cornell.edu , shayak.2015@iitkalumni.org

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Abstract

In this Letter I will investigate a simple but general model of shock waves : 'subsonic' and 'supersonic' propagation of a disturbance along a one-dimensional string satisfying the 1+1 WAVE equation. Obtaining an integral formula for the wavefront in response to the travelling disturbance, I will use it to exactly calculate the front for a simple pulse shape. Across the sound barrier, there is a phase flip of the wavefront, which does not seem to have been reported in Literature.

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It is common experience that subsonic and supersonic waves are completely different in character. For a normal passenger plane like a Boeing 777, the whine of the jets is always an indicator of its arrival while for a supersonic jet like Concorde, nothing can be heard until the jet is past and the sonic boom thereafter is deafening (it is one of the persistent obstacles in the path of supersonic travel). Standing at a harbour, we can see that the wake of a huge but crawling ocean liner spreads out in all directions remaining close to the vessel, while that of a small but energetic speedboat radiates out entirely behind the boat and comes crashing onto the shore. When particles travel through a medium faster than the speed of light in the medium (EINSTEIN forbids this only for vacuum!) they produce CHERENKOV radiation which is again produced entirely behind the particle and with greatly increased intensity.

Extensive literature exists on sonic boom, almost everything being summarized in the comprehensive Reference [1]. The primary focus areas are the jump in thermodynamic quantities across the shock. The shock is usually obtained as a solution of a nonlinear, often first order wave equation, derived from the NAVIER-STOKES equation of fluid mechanics. The form of wavefront itself is not of great interest in this context. More detailed wavefront calculations exist in the field of CHERENKOV radiation. The seminal paper by ILYA FRANK and IGOR TAMM [2], later amplified by TAMM [3], explicitly calculates the electromagnetic field of an electron moving at superluminal speed through a material medium. A difficult calculation leads to analytical determination of the 'MACH cone' with which most of us are familiar, and also to the (quoting verbatim) "absolute intensity, polarization and angular distribution of the radiation and its dependence on the index of refraction…..". A similar calculation, this time with emphasis on transition radiation from sub- to superluminal operation, was performed by GUIDO BECK [4].

In this Letter, I will try to obtain some basic features common to all shock waves – aircraft, electron, boat – by considering a simple but general model. The most basic feature of a shock is that the WAVE equation is involved in the dynamics. So here I will consider that equation in its pristine form, in one spatial (and of course one temporal) dimension. This equation models a stretched string. To create the effect of the propagating disturbance (the airliner, ship or superluminal electron) I will drag a hammer (details to follow) along the string at a speed which can be less or greater than the wave velocity. In this situation, I will calculate the wavefront explicitly. This calculation reveals a phase flip of the waves during the transition from subsonic to supersonic – a conclusion which does not seem to have been noted previously. We note that this model is different from the models of 1-dimensional shocks studied in References [5-10] and similar works. In those works, the disturbance is applied only at one end of the chain and its propagation studied – solitonic or shock solutions occur as a result of nonlinearities rather than a mismatch of speeds. Those models are not representative of the supersonic aircraft or the superluminal electron, and hence of limited relevance here.

We take our domain as an infinitely long 1-dimensional string which satisfies the linear 1+1 WAVE equation with wave velocity c. Let there be a hammer which can pluck or strike or produce any other kind of disturbance in a localized section of the string. Then, let us drag the hammer along the string at speed v so that the disturbance becomes a propagating one. Our aim will be to determine the shape of the string in the cases $v<c$ and $v>c$.

The equation for the perpendicular displacement $y(x,t)$ of string with a travelling disturbance is the inhomogeneous 1+1 WAVE equation

$$\frac{\partial^2 y}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}=h(x-vt) \quad , \tag{1}$$

where h is a function depending on the shape of the hammer. This of course splits into a homogeneous and a particular solution; if v does not equal c, the latter can be written as $Q(x-vt)$ where

$$Q''\left(1-\frac{v^2}{c^2}\right)=h \quad , \tag{2}$$

prime denoting differentiation with respect to the function argument. In other words, the particular solution also travels along the string at speed v. The homogeneous solutions can be obtained by solving the homogeneous equation (of course, but as we will see, this is easier said than done).

Let us now assume that the hammer is of the pluck variety and that, if it is applied to the undisplaced and stationary string at the point $x=0$ at time $t=0$, it produces the particular solution

$$y_p(x,t)=f(x)\ ,\ t>0 \quad , \tag{3}$$

where $f(x)$ is nonzero only in the region $-b\leq x\leq b$ ($2b$ is the width of the hammer). We will not pay special attention to the forcing function h needed to create this particular solution – let it be infinite if required, so long as the end result does not become implausible. Now we will consider this hammer to be dragged along the string at speed v, which may be less than or greater than c.

We will first solve a simpler problem than the moving hammer – that of the hammer being applied suddenly to the undisplaced and stationary string at $x=0$ and $t=0$. Of course, y_p is given by (3). To match the initial conditions, the homogeneous solution must satisfy

$$y_h(x,0)=-f(x) \quad , \tag{4a}$$

$$\dot{y}_h(x,0)=0 \quad , \tag{4b}$$

whence from D'ALAMBERT's formula,

$$y_h(x,t)=-\frac{1}{2}\left[f(x+ct)+f(x-ct)\right] \quad . \tag{5}$$

For reasons to be apparent in a moment, I will call this the '**creation solution**' and denote it by $y^+(x,t)$. The reverse situation is sudden removal of the hammer from the string. This is exactly identical to creating a 'negative hammer' so the particular and homogeneous solutions for this case will be the negatives of (3) and (4). The latter solution I will call the '**annihilation solution**' and denote by $y^-(x,t)$.

Now we will generate the solution for the moving hammer in terms of the creation and annihilation solutions. The initial condition is that at t=0 the hammer is sitting at x=0 when it starts moving to the right with speed v. First we will break up the continuous motion into a series of discrete steps. We assume that from t=0 to $t=\Delta t$ the hammer sits quietly at x=0. Then, at $t=\Delta t$, it gets annihilated at x=0 and a new hammer gets created at $x=v\Delta t$. This new hammer sits at the new location for another time interval of Δt before getting annihilated and a third one getting created at $x=2v\Delta t$. This way, in fits and starts, the hammer makes its progress down the string. The particular solution after each creation is clearly (3) with x=0 replaced by the location of the latest creation; the homogeneous solutions from each creation and annihilation event can then be added up to form the total solution $y(x,t)$. In the limit $\Delta t \rightarrow 0$ the sum will get replaced by an integral and the solution should become exact. This is very similar to what is done when calculating eddy currents using the method of retreating images [11,12] – the fundamental building block there is the creation and annihilation of a magnetic monopole and every realistic magnetic phenomenon is broken down into a succession of these basic units.

So we have got the particular solution due to the moving hammer

$$y_p(x,t) = f(x - vt) \quad , \tag{6}$$

and the formal sum of the homogeneous solutions

$$\begin{aligned} y_h(x,t) &= y^-(x,t-\Delta t) + y^+(x - v\Delta t, t - \Delta t) \\ &\quad + y^-(x - v\Delta t, t - 2\Delta t) + y^+(x - 2v\Delta t, t - 3\Delta t) + \ldots\ldots \end{aligned} \quad \text{, or} \tag{7a}$$

$$y_h(x,t) = \sum_{n=1}^{N} \left[y^-(x - \{n-1\} v\Delta t, t - n\Delta t) + y^+(x - nv\Delta t, t - n\Delta t) \right] \quad , \tag{7b}$$

where n is the summation index and N the upper limit of the sum i.e. $N\Delta t$ is the current time t at which we want to find the actual solution. The truncation of the sum at the current time imposes causality on the solution – the string cannot be influenced now by what it will do in future. Now substituting (5) and its negative for the creation and annihilation solutions,

$$y_h(x,t) = \frac{1}{2} \sum_{n=1}^{N} \begin{bmatrix} f(x + ct - n\{v + c\}\Delta t + v\Delta t) + f(x - ct - n\{v - c\}\Delta t + v\Delta t) \\ - f(x + ct - n\{v + c\}\Delta t) - f(x - ct - n\{v - c\}\Delta t) \end{bmatrix} \quad . \tag{8}$$

Performing a Taylor expansion (sweeping all issues of existence of derivative under the carpet), we have

$$y_h(x,t) = \frac{1}{2} v\Delta t \sum_{n=1}^{N} \left[f'(x + ct - n\{v + c\}\Delta t) + f'(x - ct - n\{v - c\}\Delta t) \right] \quad . \tag{9}$$

The last step is the transition from discrete to continuous – $N\Delta t$ becomes the current time t while $n\Delta t$ becomes a dummy time τ which is a variable of integration and runs from 0 to t. Thus

$$y_h(x,t)=\frac{1}{2}v\int_0^t \mathrm{d}\tau\left[f'\left(x+ct-\{v+c\}\tau\right)+f'\left(x-ct-\{v-c\}\tau\right)\right] \quad . \tag{10}$$

Equations (6) and (10) together give $y(x,t)$ in response to the moving hammer. The latter equation is very similar to the retarded GREEN's function for the WAVE equation and is the appropriate form of that function valid for the situation of moving excitation.

Now I will work out the integrals explicitly for a very simple choice of f: that of rectangular pulse of width $2b$ denoted by R where $R(s)=1$ if $-b\le s\le b$ and $R(s)=0$ otherwise. Its derivative is $\delta(s+b)-\delta(s-b)$ where $\delta(\ldots)$ denotes the DIRAC delta function. Adding (6) and (10) now gives

$$y(x,t)=R(x-vt)+\frac{1}{2}v\int_0^t \mathrm{d}\tau\begin{bmatrix}-\delta\left(x+ct-\{v+c\}\tau-b\right)+\delta\left(x+ct-\{v+c\}\tau+b\right)\\ -\delta\left(x-ct-\{v-c\}\tau-b\right)+\delta\left(x-ct-\{v-c\}\tau+b\right)\end{bmatrix} \quad . \tag{11}$$

There are four terms inside this integral which we will denote by I_1, I_2, I_3 and I_4 respectively.

What we are interested in is the motion of the string seen by an observer at some point x which is much greater than b, the size of the hammer. Since the integrals are all delta functions they will either be zero or nonzero and it remains to find out when each of them will assume the latter values. Considering I_1, let us fix a value of t. Then, at $\tau=0$ the thing inside the delta is clearly greater than zero, and it starts to decrease as τ increases. Hence, when the integral first breaks ground, the contribution will come from the uppermost limit of the τ integral i.e. $\tau=t$; at this τ the argument of delta is $x-vt-b$. This becomes zero at $t=(x-b)/v$ and the integral jumps up to $v/2(v+c)$. At all subsequent times t, the τ integral will include the delta function and the integral will continue to have this value. Analogously, the second term I_2 will turn on at $t=(x+b)/v$ and its value will be the negative of I_1. Thus, I_1 and I_2 together describe a rectangular pulse of width $2b$ travelling at speed v i.e.

$$I_1+I_2=\frac{v}{2(v+c)}R(x-vt) \quad . \tag{12}$$

The integrals I_3 and I_4 change character accordingly as v is less than or greater than c, so we need to consider it case by case.

CASE $v<c$:

Here, $v-c$ is negative. For fixed t, the argument of the delta starts off from a low value, increasing as τ goes on. The first breaking of ground will come from the lower limit of integration i.e. $\tau=0$; at this τ the argument of delta is $x-ct-b$. This becomes zero for the first time at $t=(x-b)/c$ and the integral acquires the value $v/2(v-c)$. Following the steps for I_1, we can say that

$$I_3 + I_4 = -\frac{v}{2(c-v)} R(x-ct) \quad , \tag{13}$$

where I have flipped the sign on the denominator to make the overall negative explicit. Note that this wave is travelling at speed c. ■

CASE $v>c$:

Here $v-c$ is positive. As for I_1, the first breaking ground comes from the upper limit of the τ integration; setting $\tau=t$ gives the time of $(x-b)/v$ for arrival of the pulse. Following through the rest of the steps,

$$I_3 + I_4 = \frac{v}{2(v-c)} R(x-vt) \quad . \tag{14}$$

Note that this wave is travelling at speed v not c ! ■

Now recall that the particular solution is $R(x-vt)$ and put everything together. In the subsonic case the response is

$$y(x,t) = -\frac{v}{2(c-v)} R(x-ct) + \left[1 + \frac{v}{2(v+c)}\right] R(x-vt) \quad . \tag{15}$$

Here I have assumed that the width b is small enough or equivalently the gap between v and c large enough so that the integrals turn on in the order I_3, I_4 and then I_1, I_2. The overlapping situation will create interference between the two waves and is not of great interest in this context. In the supersonic case the response is

$$y(x,t) = \frac{2v^2 - c^2}{v^2 - c^2} R(x-vt) \quad . \tag{16}$$

I will now conclude this Letter with a brief discussion.

Equations (13,14) clearly show the phase flip as we cross the sound barrier – the leading wave (13) describes a pulse which is 180° out of phase with the particular solution (6) and the trailing wave (12) while (14) describes a pulse which is in phase with the other two. This phase reversal also explains the increased energy of the supersonic shock waves relative to the subsonic waves. At v just smaller than but very nearly c, there is considerable interference between the pulses at speeds c and v; since they are out of phase, the interference is destructive, reducing the total amplitude. On the other hand, at v just greater than c, all the waves are in phase so the amplitudes add for a wave of great energy. It also follows that the results will hold for any shape of the pulse f and not just the rectangle treated explicitly – every pulse can be broken into a succession of rectangular pulses at different locations and the preceding calculations carried out for each of them to obtain the same amplitudes and phase factors in (13-16).

To forestall criticism that the shock wave model here is over-simplified and has no relation to reality, I will now show that it correctly predicts the basic sonic boom phenomena with which we are all familiar. The first such feature is that the arrival of a Boeing 777 is heralded by its whine while all the sound of Concorde comes after the aircraft is past. This is visible from (15) and (16). The subsonic wave has a pulse moving at speed c which arrives ahead of the hammer itself, moving at speed v. The supersonic hammer has everything arriving at speed v – no sound before the plane itself. The second notable feature is the increased energy of the shock wave compared to the regular wave, which I already discussed above. Finally, it is known that the sonic boom becomes weaker as the cruising speed of the aircraft increases. This is also clear from (16) – the amplitude of the shock has a maximum at v just greater than c and thereafter decreases slowly with increase in v.

So this simple but exactly solvable model can explain many features associated with shock waves in much more complicated situations. It also shows a phase reversal of the transmitted wave, which seems to have gone unnoticed so far. Although the model here is basic, the source of the phase reversal does appear very fundamental – it comes from the sign flip of $v–c$ in (10) as v crosses the sound barrier. The same sign flip causes the change in wave travel speed from c to v as the barrier is crossed. This latter phenomenon is universal and well documented, and the phase change appears to 'ride along' with it. This phenomenon can be extended by more complicated calculations involving three-dimensional waves, and suitable laboratory experiments, which are being left for the future.

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